Functional equation in the fundamental class of functions and the type of explicit formula

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Abstract

The Jorgenson-Lang fundamental class of functions encompass most of zeta and $L-$ functions of modern number theory. We highlight the role of conditions at zero and at infinity on test functions in the corresponding explicit formulas, depending on the form of a triple $(Z, \widetilde{Z}, \Phi)$ belonging to this fundamental class. The advance in explicit formulas presented here consists in enlarging the class of test functions to which these formulas can be applied.

Keywords
Jorgenson-Lang fundamental class; Weil functional; Selberg trace formula; Explicit formulas

1. Introduction

The question how the prime numbers 2, 3, 5, 7, 11, 13, 17,... are distributed in the set of all positive integers is one of the fundamental problems of arithmetic. The statement that the number of primes $\pi(x)$ less than or equal to x differs from the value of function $Li(x) = \int\limits_0^x \frac{dt}{\log t}$ by $O\left(x^{1/2}\log x\right)$ is one of equivalent forms of the most famous unsolved problem in modern mathematics - Riemann hypothesis. Riemann opens his celebrated memoir [22] with the Euler product formula

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s} = \prod_{\substack{p \\ p \text{ prime}}} (1-\frac{1}{p^s})^{-1}.$$

He complexifies the variable s (both the series and the product in the above formula converge absolutely for Re $s > 1$), derives the functional equation for $\zeta(s)$ and proceeds to express $\pi(x)$ in terms involving the zeros of ζ.

According to the explicit formula, proved by von Mangoldt,

$$\sum_{p}\sum_{n:\ p^n \le x} \log p = x - \sum_{\rho:\ \zeta(\rho)=0} \frac{x^\rho}{\rho} - \frac{\zeta'}{\zeta}(0),$$

for $x > 1$ and x not a prime power.

The prime number theorem $\pi(x) \sim Li(x)$ $(x \to \infty)$ is deduced by Hadamard and de la Vallée Poussin from the non-vanishing of the Riemann zeta function on the line Re $s = 1$.

Riemann-von Mangoldt explicit formula is a predecessor of explicit formulas in the classical number theory. We owe to A. Weil the insight that these formulas have their analogues in a more general setting. In [28], a suitably smooth test function taken over the prime powers of the number field k is expressed as the sum of its Mellin transform over the non-trivial zeros of the related Hecke $L-$function plus an analytic term at infinity, called the Weil functional.

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Selberg trace formula [23] is an explicit formula that relates the discrete and eventually continuous spectrum of the Laplace-Beltrami operator on a Fuchsian group of the first kind with the geometric information about the group. At present, it is the only available tool to analyze the structure of the spectrum of this operator, with many applications ranging from number theory to quantum physics.

A great number of zeta and $L-$functions in modern number theory apparently obey a certain formalism: they possess a representation in the form of Euler product over primes (numbers, ideals, hyperbolic classes) and satisfy a functional equation.

A very general framework for these investigations is given by Jorgenson-Lang's concept of the fundamental class.

A triple $\left(Z, \widetilde{Z}, \Phi\right)$ is in the fundamental class [14, pp. 45-46] if the following conditions are satisfied:

a) **Meromorphy.** Functions Z and $\widetilde{Z}$ are meromorphic functions of finite order;

b) **Euler Sum.** There are sequences $\{q\}$ and $\{\widetilde{q}\}$ of real numbers greater than one that depend on Z and $\widetilde{Z}$ respectively such that:

- q and $\widetilde{q}$ converge to infinity
- there exist $\sigma_0' \geq 0$ and complex numbers $c(q)$ and $c(\widetilde{q})$ such that

$$\log Z(s) = \sum \frac{c(q)}{q^s} \text{ and } \log \widetilde{Z}(s) = \sum \frac{c(\widetilde{q})}{\widetilde{q}^s}, \text{ for all Re } (s) > \sigma_0'.$$

  These series are assumed to converge uniformly and absolutely in any half plane of a form $\mathrm{Re}(s) \geq \sigma_0' + \varepsilon > \sigma_0'$.

c) **Functional Equation.**

There exist meromorphic functions C and $\widetilde{C}$ of a finite order and number σ_0, $0 \leq \sigma_0 \leq \sigma_0'$, such that

$$Z(s) C(s) = \widetilde{Z}(\sigma_0 - s) \widetilde{C}(\sigma_0 - s), \text{ or}$$

$$Z(s) \Phi(s) = \widetilde{Z}(\sigma_0 - s) \text{ for } \Phi(s) = \frac{C(s)}{\widetilde{C}(\sigma_0 - s)}.$$

Functions $C, \widetilde{C}$ and Φ are called fudge factors of the functional equation and are assumed to be of a regularized product type.

For a summary of technicalities regarding the notions of regularized products and series, regularized product type and regularized harmonic series type, as well as the definition of their reduced order, we refer to [14, pp. 14-19 and 36-37].

The key role of the functional equation is clearly expressed in the generalization of Cramer's theorem [12], saying essentially that on horizontal lines Z and $\widetilde{Z}$ asymptotically behave like Φ.

In Section 2 we enlarge the class of test functions in the Jorgenson-Lang explicit formula. Section 3 deals with a previously uncovered case of the factor of a functional equation having infinitely many zeros or poles in the critical strip. One of the results of Section 4 is that the Selberg trace formula, when interpreted as an explicit formula, is valid for a larger class of test functions than in [23]. The possibility to choose a regularized test function with finitely many discontinuities (even at zero) is used again in Section 5. The Selberg class case is treated in Section 6. A detailed version with complete proofs of our results can be found in [3] - [6] and [25].

2. The explicit formula for the fundamental class of functions. Case 1

Let $\left(Z,\widetilde{Z},\Phi\right)$ be a triple in the fundamental class. Throughout this paper we assume that the factor Φ is of the regularized product type of order (M,m). This implies that

$$\frac{\Phi'}{\Phi}(\sigma\pm iT)=O\left(T^{M}\log^{m}T\right),\ \text{ as }\ T\to\infty,$$

uniformly in σ, $-a\leq\sigma\leq\sigma_0+a$.

We will denote by f and F test functions, related by $f(x)=F(-\log x)$, $x>0$.

Let $a>0$ be such that $\sigma_0'<\sigma_0+a$, and such that $Z,\widetilde{Z}$ and Φ do not have a zero or a pole on the lines $\mathrm{Re}\,(s)=-a$ and $\mathrm{Re}\,(s)=\sigma_0+a$.

One denotes by:

- $\{\rho\}$ the set of zeros and poles of $Z(s)$ in the full strip $-a\leq\mathrm{Re}s\leq\sigma_0+a$.
- $\{\kappa\}$ the set of zeros and poles of Φ in the half strip $-a\leq\mathrm{Re}s\leq\frac{\sigma_0}{2}$.
- $v(\rho)$ resp. $v(\kappa)$ the order of ρ resp. κ (taken with negative sign in case of a pole).

We pose the following conditions on a test function F:

C I. F is M times differentiable, $F^{(M)}\in L^1(\mathbb{R})\cap\phi BV(\mathbb{R})$
$F^{(j)}\in L^1(\mathbb{R})\cap HBV(\mathbb{R})$ for $j=0,...,(M-1)$.

C II. $F^{(M)}(x)-F^{(M)}(0)=O\left((\log|x|)^{-\alpha}\right)$, as $x\to0$, for some $\alpha>M+2$.

C III. a) There exists $a'>a$ such that $F^{(M)}(x)\,e^{\left(a'+\frac{\sigma_0}{2}\right)|x|}\in\phi BV(\mathbb{R})\cap L^1(\mathbb{R})$,
b) $F^{(j)}(x)\,e^{\left(a'+\frac{\sigma_0}{2}\right)|x|}\ll C$, $j=1,...,M-1$ (if $M\geq2$).
c) $F(x)\,e^{\left(a'+\frac{\sigma_0}{2}\right)|x|}\in HBV(\mathbb{R})\cap L^1(\mathbb{R})$

Here, ϕ is a continuous, strictly increasing convex function on $[0,\infty)$ satisfying two asymptotic conditions

(0_1) $\quad\lim\limits_{x\to0^+}\frac{\phi(x)}{x}=0,$
(∞_1) $\quad\lim\limits_{x\to\infty}\frac{\phi(x)}{x}=\infty$

and the condition that ensures the existence of the generalized Riemann-Stieltjes integral

(p) $\quad\sum\left(\frac{1}{n}\right)^{\frac{1}{p}}\phi^{-1}\left(\frac{1}{n}\right)<\infty$, for some $p>1$.

By ϕBV we denote the set of functions of bounded $\phi-$variation in the sense of L. C. Young. A function f is said to be of ϕ bounded variation on an interval I with the end points a and b if

$$V_\phi(f,I)=\sup\sum_n\phi\left(\left|f\left(I_n\right)\right|\right)<\infty,$$

where $f(I)$ stands for $f(b)-f(a)$ and the supremum is taken over all systems $\{I_n\}$ of nonoverlapping subintervals of I.

HBV denotes the Waterman class of functions of harmonic bounded variation: $f\in HBV(I)$ if

$$\sum\frac{\left|f\left(I_n\right)\right|}{n}<\infty$$

for every choice of nonoverlapping intervals $I_n \subset I$.

In a certain sense, HBV is an optimal framework for everywhere pointwise convergence results in the classical Fourier analysis in one variable, see [1].

The following theorem extends the Jorgenson-Lang explicit formula to a wider class of test functions.

Theorem 2.1 [3, Th. 6.1] *Let* $\left(Z, \widetilde{Z}, \Phi\right)$ *be in the fundamental class and assume that* Φ *is of reduced order* (M, m) *with finitely many zeros and poles in the half strip* $-a \leq \mathrm{Re} s \leq \frac{\sigma_0}{2}$ *and no zeros or poles on the line* $\mathrm{Re}\,(s) = \frac{\sigma_0}{2}$. *Then for any function* F *that satisfies conditions* C I-C III *the formula*

$$\sum_{\rho} v(\rho)\, M_{\frac{\sigma_0}{2}} f(\rho) + \sum_{\kappa} v(\kappa)\, M_{\frac{\sigma_0}{2}} f(\kappa) = \tag{1}$$
$$= \sum_{q} \frac{-c(q)\log q}{q^{\frac{\sigma_0}{2}}} f(q) + \sum_{\widetilde{q}} \frac{-c(\widetilde{q})\log \widetilde{q}}{\widetilde{q}^{\frac{\sigma_0}{2}}} f\left(\frac{1}{\widetilde{q}}\right) + W_{\Phi}(F)$$

holds true.

The sum on the left is taken in the sense of the limit as $\mathrm{Im}\rho \to \infty$, and $M_{\frac{\sigma_0}{2}} f$ denotes the translate by $\frac{\sigma_0}{2}$ of the Mellin transform of the function f.

Remark 2.1 Growth conditions C I and C II are related to the evaluation of the Weil functional

$$W_{\Phi}(F) = \lim_{T\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-T}^{T} \widehat{F}(t)\, \frac{\Phi'}{\Phi}\left(\frac{\sigma_0}{2} + it\right) dt.$$

As proved in [2], the Weil functional is well defined for test functions satisfying the above assumptions. CI and CII are less restrictive than Jorgenson-Lang's conditions [13, p. 105]:

JL1 $F^{(j)} \in BV(\mathbb{R}) \cap L^1(\mathbb{R})$, $j = 0, ..., M$ and

JL2 There exists $\varepsilon > 0$ such that $F^{(j)}(x) - F^{(j)}(0) = O(|x|^{\varepsilon})$, as $x \to 0$, $j = 0, .., M$.

Remark 2.2 The condition CIII, relevant for the Mellin transform estimates, is also weaker than the respective

JL3 $F^{(j)}(x)\, e^{\left(\frac{\sigma_0}{2}+a'\right)|x|} \in BV(\mathbb{R}) \cap L^1(\mathbb{R})$, $j = 0, ...M$,

posed by Jorgenson and Lang in [14]. New bounds for the Mellin transform of a test function that we obtained in [3, Th. 4.1] rely on L. C. Young's results on the Stieltjes integral in the generalized Moore-Pollard sense (see, e.g. [2, Section 2]).

3. The explicit formula for the fundamental class of functions. Case 2

The implicit assumption in the proof of explicit formula in [14] is that the fudge factor Φ of the functional equation has finitely many zeros and poles in the left half $-a \leq \mathrm{Re} s \leq \frac{\sigma_0}{2}$ of the critical strip. However, if this is not the case, the sum over κ on the left hand side of the explicit formula (1) need not converge. It might not be possible to move the line of integration to the right and obtain the Weil functional. Therefore, we consider a generalized version

$$W_{\Phi,a}(F) = \lim_{T\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-T}^{T} \widehat{F}(t)\, \frac{\Phi'}{\Phi}(a + it)\, dt.$$

Additionally, we pose the Orlicz condition on ϕ that turns ϕBV into a linear space:

(Δ_2) there exist positive constants x_0 and d ($d \geq 2$) such that $\phi(2x) \leq d\phi(x)$ for all $0 \leq x \leq x_0$.

The setting being more general than in the previous section, we require that an M times differentiable test function F satisfies:

- Ex I $\qquad F^{(j)}(x)\, e^{\left(a'+\frac{\sigma_0}{2}\right)|x|} \in \phi BV(\mathbb{R}) \cap L^1(\mathbb{R})$,
- Ex II $\qquad F^{(j)}(x) = F^{(j)}(0) + O\left(|\log|x||^{-\alpha}\right) \qquad (\alpha > M+2)$,

for some $a' > a > 0$ and $j \in \{0, 1, ...M\}$.

The following theorem gives us a general explicit formula.

Theorem 3.1 [5, Th. 5.1] *Let* $\left(Z, \widetilde{Z}, \Phi\right)$ *be in the fundamental class of functions and assume that* Φ *is of a reduced order* (M, m). *Then, for a test function* F *fulfilling conditions* Ex I *and* Ex II, *the formula*

$$\sum_\rho v(\rho)\, M_{\frac{\sigma_0}{2}} f(\rho) = \sum_q \frac{-c(q)\log q}{q^{\frac{\sigma_0}{2}}} f(q) + \sum_{\widetilde{q}} \frac{-c(\widetilde{q})\log \widetilde{q}}{\widetilde{q}^{\frac{\sigma_0}{2}}} f\left(\frac{1}{\widetilde{q}}\right) + W_{\Phi,-a}(F_a)$$

holds true, where $F_a(x) = F(x)\, e^{\left(\frac{\sigma_0}{2}+a\right)x}$.

Remark 3.1 Note that conditions Ex I and Ex II imply CI-CIII.

The evaluation of the sums appearing in the above theorem is done in the same way as in the proof of Theorem 2.1: by a refined contour integration, application of the functional equation and improved Mellin transform estimates. Conditions Ex I and Ex II, combined with (Δ_2) imply that F_a satisfies C I and C II. Therefore, it is possible to apply the Fourier inversion theorem [2, Th. 3.1] and a general Parseval formula [3, Lemma 8.3] to the function F_a and obtain the existence and evaluate the generalized Weil functional.

4. Symmetric case

In many applications of the explicit formulas, functions Z and $\widetilde{Z}$ are equal. We refer to this case as a symmetric one, since the functional equation $Z(s)\Phi(s) = Z(\sigma_0 - s)$ implies that the zeros of Z are symmetric with respect to the line $\mathrm{Re}(s) = \frac{\sigma_0}{2}$ and the equality $\frac{\Phi'}{\Phi}\left(\frac{\sigma_0}{2}+it\right) = \frac{\Phi'}{\Phi}\left(\frac{\sigma_0}{2}-it\right)$ holds for almost all t (except eventually at zeros $\frac{\sigma_0}{2}+it$ and $\frac{\sigma_0}{2}-it$ of Z). The last property is essential for a simpler evaluation of the Weil functional, since it allows us to write the latter in the form

$$\begin{aligned} W_\Phi(F) &= \frac{1}{\sqrt{2\pi}} \lim_{T\to\infty} \int_{-T}^{T} \widehat{F}(t)\, \frac{\Phi'}{\Phi}\left(\frac{\sigma_0}{2}+it\right) dt \\ &= \frac{1}{2\sqrt{2\pi}} \lim_{T\to\infty} \int_{-T}^{T} \left(\widehat{F}(t) + \widehat{F}(-t)\right) \frac{\Phi'}{\Phi}\left(\frac{\sigma_0}{2}+it\right) dt. \end{aligned}$$

Now, the proof of the explicit formula can be modified replacing the function $F(x)$ by $G(x) = F(x) + F(-x)$.

Moreover, in [4, Th. 3.1] it is shown that a Parseval formula in the symmetric case holds with a test function F that is subject to less restrictive differentiability conditions.

Theorem 4.1 *Let F be an $(M-1)$ times differentiable function such that $F^{(M-1)}$ is differentiable everywhere except at finitely many points where it has a left and a right derivative. At a point x where the function $F^{(M-1)}$ is not differentiable, we define $F^{(M)}(x) = \frac{1}{2}\left(F^{(M)}(x+0) + F^{(M)}(x-0)\right)$. Let the function $G(x) = F(x) + F(-x)$ satisfy conditions* CI.-CIII. *of Theorem 2.1. Then, for the triple (Z, Z, Φ) in the fundamental class, where Φ is of reduced order (M, m) with no zeros or poles on the line* $\operatorname{Re}(s) = \frac{\sigma_0}{2}$, *the explicit formula*

$$\lim_{T\to\infty}\left(\sum_{\substack{-a\le\operatorname{Re}(\rho)\le\sigma_0+a,\\ |\operatorname{Im}\rho|\le T}} v(\rho)\cdot M_{\frac{\sigma_0}{2}}f(\rho) + \sum_{\substack{-a\le\operatorname{Re}(\kappa)\le\frac{\sigma_0}{2},\\ |\operatorname{Im}\kappa|\le T}} v(\kappa)\cdot M_{\frac{\sigma_0}{2}}f(\kappa)\right) \tag{2}$$
$$= \sum_q \frac{-c(q)\log q}{q^{\frac{\sigma_0}{2}}}\left(f(q) + f\left(\frac{1}{q}\right)\right) + W_\Phi(F)$$

holds.

If $M = 1$, the assumption on $F = F^{(0)}$ in Theorem 3.1 is that F is continuous. If $M = 0$, the assumption on F is that it is a regularized function, i.e. F possesses one-sided limits at each point and $F(x) = \frac{1}{2}(F(x+0) + F(x-0))$.

Remark 4.1 Theorem 4.1 is a generalization of the explicit formula [4, Th. 4.1] to the case when the factor Φ of the functional equation has infinitely many zeros or poles in the left half of the critical strip. The point is in the existence of the limit on the left hand side of (2), though the limits of the two sums considered separately might not exist. The modification in the proof comes from the fact that $Z = \widetilde{Z}$ enables us to avoid the evaluation of the generalized Weil functional. This is done by careful contour integration, application of the functional equation and through consideration of the function $G(x) = F(x) + F(-x)$ instead of F.

Under Riemann hypothesis, application of Theorem 4.1 with $M = 0$ to the triple (ζ, ζ, η) and the test function F_y defined by

$$F_y(x) = \begin{cases} e^{-\alpha x}, & x \ge -y \\ e^{\alpha(x+2y)} & x < -y \end{cases}, \ \operatorname{Re}\ (\alpha) > \frac{1}{2},\ \ y > 0, \tag{3}$$

yields by a meromorphic continuation a new representation of the logarithmic derivative of the Riemann zeta function and a better conditional estimate for its growth [4], given by

Theorem 4.2
a) For Re $\alpha > 0$,

$$\frac{\zeta'}{\zeta}\left(\frac{1}{2}+\alpha\right) = \frac{-1}{1+e^{2y\alpha}}\sum_{p^n\le e^y}\frac{\log p}{p^{n\left(\alpha+\frac{1}{2}\right)}}\left(e^{2y\alpha} - p^{2n\alpha}\right) - \frac{4\alpha e^{y\alpha}\cosh\frac{y}{2}}{\left(1+e^{2y\alpha}\right)\left(\alpha^2-\frac{1}{4}\right)}$$
$$+\frac{4\alpha e^{y\alpha}}{1+e^{2y\alpha}}\left[\frac{N(0)}{\alpha^2} + \int_0^\infty \cos ty\frac{d\left(S(t)+E(t)\right)}{\alpha^2+t^2} - \frac{1}{2\pi}\int_0^\infty\frac{\cos ty\cdot H(t)\,dt}{\left(s-\frac{1}{2}\right)^2+t^2}\right],$$

b)

$$\frac{\zeta'}{\zeta}(s) = O\left(\min\left\{\frac{\log T}{\left(\sigma-\frac{1}{2}\right)\log\log T}, \frac{\left(\frac{\log T}{\sigma-\frac{1}{2}}\right)^{2-2\sigma}-1}{e^{1-\sigma}-1}\right\}\right)$$

as $T\to\infty$, for $s = \sigma + iT$ and $2 > \sigma > \frac{1}{2} + \frac{1}{\log T}$.

Here, $N(t)$ denotes the number of zeros $\rho = \frac{1}{2} + i\gamma$ of ζ such that $0 < \gamma \le t$, functions $S(t)$ and $E(t)$ are related through $dN(t) = \frac{1}{2\pi} \log \frac{t}{2\pi} dt + d(S(t) + E(t))$ (see [26, Th. 9.3 on p. 212]) and

$$H(t) = \frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{it}{2}\right) + \frac{\Gamma'}{\Gamma}\left(\frac{1}{4} - \frac{it}{2}\right) - 2\log\frac{t}{2}.$$

Remark 4.2 Theorem 4.2 b) improves the estimate given in [26, p. 383].

Application of Theorem 4.1 with $M = 1$ to the triple $(Z_\Gamma, Z_\Gamma, \Psi_\Gamma)$, where

$$Z_\Gamma(s;\chi) = \prod_{\{P_0\}_\Gamma} \prod_{k=0}^{\infty} \left(1 - \chi(P_0) \cdot N(P_0)^{-s-k}\right),$$

is the Selberg zeta function of a strictly hyperbolic cocompact Fuchsian group Γ with a fixed unitary representation χ, and Ψ_Γ the factor of the functional equation $Z_\Gamma(s)\Psi_\Gamma(s) = Z_\Gamma(1-s)$, yields the following theorem.

Theorem 4.3 *Let F be a continuous, integrable function on $\mathbb{R}$ such that F' exists everywhere except at finitely many points where F has a right and a left derivative. At a point x where the function F is not differentiable, we define $F'(x)$ as $F'(x) = \frac{1}{2}(F'(x+0) + F'(x-0))$. We assume that the function $G(x) = F(x) + F(-x)$ and its derivative satisfy the following two conditions:*

1. $G^{(j)}(x) e^{\left(\frac{1}{2}+\varepsilon\right)|x|} \in \phi BV(\mathbb{R}) \cap L^1(\mathbb{R})$ *for some* $\varepsilon > 0$, $j = 0, 1$.
2. $G'(x) - G'(0) = O\left(|\log|x||^{-\alpha}\right)$ *for* $x \to 0$ $(\alpha > 3)$.

Then, the Selberg trace formula

$$\sum_j h(r_j) = \frac{|\mathfrak{F}|}{2\pi} \int_{-\infty}^{\infty} r h(r) \tanh(\pi r)\, dr + \tag{4}$$

$$+\frac{2}{\sqrt{2\pi}} \sum_{\{P_0\}_\Gamma} \sum_{n=1}^{\infty} \frac{\chi^n(P_0) \log N(P_0)}{N(P_0)^{n/2} - N(P_0)^{-n/2}} \widehat{h}(n \log N(P_0))$$

holds for the function

$$h(u) := \frac{1}{2} \int_{-\infty}^{\infty} G(x) e^{-iux} dx,$$

defined in the strip $|\mathrm{Im}\ u| < \frac{1}{2} + \varepsilon$.

The sum on the left hand side of (4) is taken over all solutions of the equation $\frac{1}{4} + r_j^2 = \lambda_j$, where λ_j are eigen-values of the (unique self-adjoint extension of) Laplace-Beltrami operator on $L^2(\Gamma\backslash\mathcal{H}, d\mu, \chi)$. The sum on the right hand side is taken over the set of all primitive hyperbolic conjugacy classes of Γ and $N(P)$ is the norm of the elements in the Γ-conjugacy class $\{P\}_\Gamma$ of the hyperbolic element $P \in \Gamma$.

The proof of Theorem 4.3 can be found in [4, Section 6].

Remark 4.3 The function $h_y(u) = \frac{2\alpha e^{y\alpha}}{\alpha^2+u^2} \cos yu$, $|\mathrm{Im}\ u| < \mathrm{Re}\ \alpha$ which corresponds to the test function (3) does not satisfy Selberg's third condition (i.e. it is not bounded by $\left(1+|u|^2\right)^{-(1+\epsilon)}$), see [23], [27]. Therefore, according to Theorem 4.3, the Selberg trace formula holds for a wider class of test functions.

Remark 4.4 In the case of a non-compact Fuchsian group of the first kind, the factor Ψ_Γ of the functional equation satisfied by the corresponding Selberg zeta function has infinitely many poles in the critical strip (see, e.g. [10, pp. 498-500]). Since Theorem 4.1 covers this case as well, the trace formula [10, Th. 6.3 on p. 412] applies also to the test function h of Theorem 4.3.

5. Hyperbolic scattering determinant

Millson-Shintani zeta function [18] and hyperbolic scattering determinant [15] are examples of functions in the fundamental class that satisfy the equation of the form $Z\left(s\right)Z\left(\sigma_0-s\right)=C$.

Due to $\widetilde{Z}=\frac{1}{Z}$, we may consider requirements on $\left(F\left(x\right)-F\left(-x\right)\right)$ instead on F. Furthermore, the fudge factor of the functional equation being a constant, it is possible to prove an explicit formula without posing a condition at zero on a test function.

Theorem 5.1 *Let* $\left(Z,\frac{1}{Z},C\right)$ *be a triple in the fundamental class, where* C *is a non-zero constant. Then*

$$\sum_{\rho}v\left(\rho\right)M_{\frac{\sigma_0}{2}}g\left(\rho\right)=\sum_{q}\frac{c\left(q\right)\log q}{q^{\frac{\sigma_0}{2}}}G\left(\log q\right)$$

holds for a regularized test function F *such that*

$$\left(F\left(x\right)-F\left(-x\right)\right)e^{\left(\frac{\sigma_0}{2}+a'\right)\left|x\right|}\in\phi BV\left(\mathbb{R}\right)\cap L^1\left(\mathbb{R}\right),$$

where $G\left(x\right)=F\left(x\right)-F\left(-x\right)$.

The hyperbolic scattering determinant for a Fuchsian group $\Gamma\subseteq PSL\left(2,\mathbb{R}\right)$ of the first kind with n_1 parabolic subgroups can be represented as $\psi\left(s\right)=K\left(s\right)H\left(s\right)$, where $K\left(s\right)=\pi^{\frac{n_1}{2}}\left(\frac{\Gamma\left(s-\frac{1}{2}\right)}{\Gamma\left(s\right)}\right)^{n_1}e^{c_1s+c_2}$, for some constants c_1 and c_2 depending upon Γ, and $\frac{H'}{H}\left(s\right)=\sum\limits_{q}\frac{b\left(q\right)}{q^s}$, with the series on the right converging absolutely and uniformly for $\mathrm{Re}s\geq a+\epsilon$ ($a\geq 1$ and $\epsilon>0$), see [10]. It satisfies the functional equation $\psi\left(s\right)\psi\left(1-s\right)=1$.

Application of Theorem 5.1 to $\psi\left(s\right)=K\left(s\right)H\left(s\right)$, yields the following theorem.

Theorem 5.2 [6, Th. 1.1] *If* F *is a regularized function such that*

$$\left(F\left(x\right)-F\left(-x\right)\right)e^{\left(\frac{1}{2}+a'\right)\left|x\right|}\in\phi BV\left(\mathbb{R}\right)\cap L^1\left(\mathbb{R}\right)$$

for some $a'>a$, *then*

$$\sum_{\rho}v\left(\rho\right)M_{\frac{1}{2}}g\left(\rho\right)=-\sum_{q}\frac{b\left(q\right)}{q^{\frac{1}{2}}}G\left(\log q\right)+n_1\int_0^\infty\frac{e^{\frac{x}{2}}G\left(x\right)}{1+e^{\frac{x}{2}}}dx. \tag{5}$$

The above theorem shows that the explicit formula for the hyperbolic scattering determinant [15, Th. 7.1] holds for a larger class of test functions. The integral on the right hand side of (5) arises from the factor $K\left(s\right)$.

Application of Theorem 5.2 to the test function

$$F\left(x\right)=\begin{cases}e^{-\left(s-\frac{1}{2}\right)x} & x>0\\ \frac{1}{2} & x=0\\ 0 & x<0\end{cases},\quad \mathrm{Re}\ s>1,$$

yields by a meromorphic continuation the following corollary.

Corollary. For all complex s distinct from zeros and poles of ψ, one has

$$\frac{\psi'}{\psi}(s) = c_1 + \sum_{\rho} \frac{v(\rho)}{s-\rho}.$$

6. An explicit formula for the Selberg class

The Selberg class of functions was introduced by A. Selberg in [24], as a class of Dirichlet series F satisfying the following conditions:

i) F possesses a Dirichlet series representation

$$F(s) = \sum_{n=1}^{\infty} \frac{a_F(n)}{n^s},$$

that converges absolutely for Re $s > 1$.

ii) There exists an integer $m \geq 0$ such that $(s-1)^m F(s)$ is an entire function of finite order. The smallest such number is denoted by m_F and called a polar order of F.

iii) The function F satisfies the functional equation

$$F(s)\, w Q_F^{2s-1} \prod_{j=1}^{r} \frac{\Gamma(\lambda_j s + \mu_j)}{\Gamma\left(\lambda_j (1-s) + \overline{\mu}_j\right)} = \overline{F(1-\overline{s})} = \overline{F}(1-s),$$

with $Q_F > 0$, $r \geq 0$, $\lambda_j > 0$, $|w| = 1$, Re $\mu_j \geq 0$, $j = 1, ..., r$.

Though the numbers λ_1, ..., λ_r are not unique, it can be shown (see, e.g. [20]) that the number $d_F = 2\sum_{j=1}^{r} \lambda_j$ is an invariant, called the degree of F.

iv) (Ramanujan conjecture) For every $\epsilon > 0$, $a_F(n) << n^{\epsilon}$.

v) (Euler product)

$$\log F(s) = \sum_{n=1}^{\infty} \frac{b_F(n)}{n^s},$$

where $b_F(n) = 0$, for all $n \neq p^m$ with $m \geq 1$ (p is a prime) and $b_F(n) << n^{\theta}$, for some $\theta < 1/2$.

An extended Selberg class $\mathcal{S}^{\#}$ is a class of functions satisfying conditions (i), (ii) and (iii).

A very nice introduction into theory of the Selberg class and extended Selberg class can be found in surveys [20] and [21]. It is believed that this class coincides with the class of all automorphic $L-$functions. So, the theory of the Selberg class represents an analytic counterpart of the Langlands program.

For some applications, it is not necessary to pose multiplicativity and boundedness conditions on the coefficients $a_F(n)$ and $b_F(n)$, as in v). Therefore, we introduce the following Euler sum condition as a weaker alternative.

v') (Euler sum) The logarithmic derivative of the function F possesses a Dirichlet series representation

$$\frac{F'}{F}(s) = -\sum_{n=1}^{\infty} \frac{c_F(n)}{n^s},$$

converging absolutely for Re $s > 1$.

The condition v') and the functional equation iii) imply that all zeros of F, except the trivial ones (i.e., zeros $\rho = -\frac{\mu_j + k}{\lambda_j}$, $k = 0, 1, ...$ and $j = 1, ..., r$) lie in the critical strip $0 \leq \mathrm{Re} s \leq 1$.

The extended Selberg class is an important special case of the fundamental class of functions, with $M = 0$, if we consider the former one as a set of triples $\left(F, \overline{F}, \Psi_F\right)$, with the factor of the functional equation given by

$$\Psi_F = wQ_F^{2s-1}\prod_{j=1}^{r}\frac{\Gamma\left(\lambda_j s + \mu_j\right)}{\Gamma\left(\lambda_j\left(1-s\right)+\overline{\mu}_j\right)}$$

and F satisfying the condition v'). Then, the application of our results on explicit formulas yields the following theorem:

Theorem 6.1 *Let a regularized function G fulfill the following conditions:*
1. $G \in \phi BV\left(\mathbb{R}\right) \cap L^1\left(\mathbb{R}\right)$
2. $G\left(x\right) e^{(1/2+\epsilon)|x|} \in \phi BV\left(\mathbb{R}\right) \cap L^1\left(\mathbb{R}\right)$, *for some* $\epsilon > 0$.
3. $G\left(x\right) + G\left(-x\right) - 2G\left(0\right) = O\left(\left|\log\left|x\right|\right|^{-\alpha}\right)$, *as* $x \to 0$, *for some* $\alpha > 2$.

Then, the formula

$$\lim_{T\to\infty}\sum_{\substack{0\leq \mathrm{Re}\rho\leq 1\\ |\mathrm{Im}\rho|\leq T,\ \rho\neq 0}} ord\left(\rho\right)M_{\frac{1}{2}}g\left(\rho\right) = m_F M_{\frac{1}{2}}g\left(0\right) + m_F M_{\frac{1}{2}}g\left(1\right) -$$

$$-\sum_{n}\frac{c_F\left(n\right)}{n^{1/2}}g\left(n\right) - \sum_{n}\frac{\overline{c_F}\left(n\right)}{n^{1/2}}g\left(1/n\right) + 2G\left(0\right)\log Q_F +$$

$$+\sum_{j=1}^{r}\int_0^{\infty}\left[\frac{2\lambda_j G_j\left(0\right)}{x} - \frac{\exp\left(\left(1-\frac{\lambda_j}{2}-\mathrm{Re}\mu_j\right)\frac{x}{\lambda_j}\right)}{1-e^{-\frac{x}{\lambda_j}}}\left(G_j\left(x\right)+G_j\left(-x\right)\right)\right]e^{-\frac{x}{\lambda_j}}dx$$

holds for an arbitrary function $F \in \mathcal{S}^{\#}$ satisfying condition (v').

Here, we put $g\left(x\right) = G\left(-\log x\right)$, for $x > 0$ and $G_j\left(x\right) = G\left(x\right)\exp\left(\frac{ix\mathrm{Im}\mu_j}{\lambda_j}\right)$. The sum on the left is taken over all non-trivial zeros of F.

Applying the explicit formula to the function

$$G_{n,z}\left(x\right) = \begin{cases} e^{-(z+x/2)}\sum_{l=1}^{n}\binom{n}{l}\frac{(-1)^{l-1}x^{l-1}}{(l-1)!}, & \text{if } x > 0 \\ n/2, & \text{if } x = 0 \\ 0, & \text{if } x < 0 \end{cases} \tag{6}$$

where z is a positive constant, and using the results of Bombieri and Lagarias [8, Lemma 1], one easily sees that the series

$$\lambda_F\left(n\right) = \sum_{\rho}\left[1-\left(1-\frac{1}{\rho}\right)^n\right],$$

over non-trivial zeros of $F \in S^{\#}$ (satisfying condition (v')) converges absolutely for $n \geq 2$, and in *-sense (i.e. as the limit of sums over $|\mathrm{Im}\rho| \leq T$, as $T \to \infty$) in the case $n = 1$.

Constants $\lambda_F\left(n\right)$ are analogues of Li's constants for the Riemann zeta function, defined in [17], where a positivity criterion $\lambda_\zeta\left(n\right) \geq 0$ $\left(n \in \mathbb{N}\right)$ for the Riemann hypothesis was proved. Once the convergence is established, it is possible to prove the Li's criterion for the Riemann hypothesis for the Selberg class as well.

Proposition 6.2 [25] *Let $F \in \mathcal{S}^{\#}$ be a function satisfying v') and such that* 0 *is not a zero of* F. *Then, all non-trivial zeros of* F *lie on the line Re* $s = \frac{1}{2}$ *if and only if* $\operatorname{Re} \lambda_F(n) \geq 0$ *for all* $n \in \mathbb{N}$.

Moreover, the application of Theorem 6.1 with the test function (6) enables one to express Li's constants $\lambda_F(-n) = \overline{\lambda_F(n)}$ in terms of coefficients $\gamma_F(k)$, $k = 0, ..., n-1$ appearing in the Laurent series expansion of $\frac{F'}{F}$ around its possible pole $s = 1$.

Theorem 6.3 [25] *Let $F \in \mathcal{S}^{\#}$ be a function having an Euler sum and such that* 0 *is not a zero of* F. *Then, for all* $n \in \mathbb{N}$

$$\lambda_F(-n) = m_F + n \log Q_F + \sum_{l=1}^{n} \binom{n}{l} \gamma_F(l-1) + \sum_{l=1}^{n} \binom{n}{l} \eta_F(l-1),$$

where

$$\eta_F(0) = \sum_{j=1}^{r} \lambda_j \frac{\Gamma'}{\Gamma}(\lambda_j + \mu_j) \qquad \text{and} \qquad \eta_F(l) = \sum_{j=1}^{r} (-\lambda_j)^l \sum_{k=0}^{\infty} \frac{1}{(\lambda_j + \mu_j + k)^l},$$

for $l \geq 1$.

Remark 6.1 Similar results about Li's criterion and Li's constants are attempted in [19, Corollaire 2.2 and Théorème 2.3] by different methods. However, [19, Corollaire 2.2] is incorrect. The extended Selberg class $\mathcal{S}^{\#}$ does not seem to be an appropriate setting for Corollaire 2.2 type of assertion, since the very notion of non-trivial zeros becomes questionable without Euler product assumption v) or Euler sum v'). The constants $\lambda_F(n)$ for $F \in \mathcal{S}$ need not be real numbers. For the same reason, $\lambda_F(n)$ on the left-hand side of the formula (3) in the statement of [19, Th. 2.3] should be replaced by $\lambda_F(-n)$. The logarithmic derivative in the first line of the proof should be corrected accordingly. Moreover, the proof itself suffers from two serious omissions: a) the prime number theorem for the Selberg class is equivalent to non-vanishing of $F \in \mathcal{S}$ on the line Re $s = 1$ (see [16]) b) even with this additional assumption, the use of [11, Th. 1] is not legitimate since the error term in the prime number theorem is $o(x)$ as $x \to \infty$, and [11, Th. 1] requires $O(x^b)$ for some $0 < b < 1$. (As a matter of fact, it is still not known how to get the bound $O(x^b)$ without a zero-free region for F.)

Remark 6.2 Since we obtained the arithmetic expression for $\lambda_F(n)$ using the explicit formula and not referring to the prime number theorem, Ramanujan conjecture nor growth conditions on coefficients in the Euler product v), our results apply to the automorphic $L-$functions unconditionally.

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