

Basic Problems of the Calibration of Measuring Systems Intended for Dynamic Measurements

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Abstract

The paper refers to the signals maximizing integral square error and absolute error if the constraints of magnitude as well as magnitude and rate of change are simultaneously imposed on them. The problem of existence, attainability and shape of such signals is analysed in detail in this paper. Determination of signals maximizing dynamic error criteria presents the basic problem in the process of calibration of measuring systems intended for measurement of dynamic signals.

Keywords

Measuring systems; Dynamic signals; Maximum errors; Calibration

1. Introduction

Depending on the type of input signal, measurements can be classified in a number of different ways. One way is to divide input signals into static and dynamic. Static signals are measured by means of instruments having precisely described accuracy. This accuracy is verified using calibration processes laid down by international regulations which are controlled in each country by the corresponding weights and measurements office. These regulations cover the hierarchy of standards, the calibration circuits and valid calibrating procedures. The class index resulting from the value of the maximum static errors constitutes a basic criterion of the metrological quality estimation for such instruments. It determines the classification of the calibrated instrument at an adequate level in the hierarchy of its accuracy. Mathematical methods of static errors calculation and calibrating procedures for instruments intended for static measurements are well established and have been available for a long time.

Things are very different in the case of measuring systems for dynamic measurements for which the input signal is time dependent, of arbitrary shape and impossible to predict, as for example, signals originating from seismic waves, signals of acoustic pressure generated by explosions of explosive charges, signals from earth crust vibrations caused by earthquakes, etc. For systems of this type neither legal regulations concerning calibration nor specific calibration procedures have been worked out so far. Consequently, neither dynamic accuracy classes nor the accuracy hierarchies based on them have been determined for these systems. The reason for this is that the determination of errors is possible only if the input signal used for their calculation is known. As it is impossible to analyse the full set of all imaginable input dynamic signals we need to limit the signals considered. The question immediately arises: what signals should we use for the calibration of systems which have at their input dynamic signals of unknown shape and unknown spectral distribution. A solution to this question is to consider the signal that maximizes the errors. The great merit of using such a signal is that results can be mutually compared regardless of the measured signals shape as much as the errors determined will always be greater or, at least, equal to the value resulting from a signal of any shape which could appear

at the input of the system. Effectively all the possible input signals to a real system are taken into consideration at the same time. Therefore the value of maximum errors can create the basis for the hierarchy of dynamic accuracy, just like class indexes create the basis for hierarchies of accuracy of the instruments applied for static measurements.

The paper presents chosen solutions referring to the existence and attainability of signals maximizing the integral square error and the absolute value of error with one and two constraints imposed on them. These constraints refer to magnitude as well as to maximum rate of signal change. The last constraint is applied in order to match the dynamic properties of the signal to the dynamic properties of the calibrated system.

General assumption

Let the mathematical model of a measuring system be given by state equation

$$\begin{aligned} \dot{x}_m(t) &= A_m x(t) + B_m u(t) & x_m(0) &= 0 \\ y_m(t) &= C_m^T x(t) \end{aligned} \quad (1.1)$$

and the system constituting its standard be given by a similar equation

$$\begin{aligned} \dot{x}_r(t) &= A_r x(t) + B_r u(t) & x_r(0) &= 0 \\ y_r(t) &= C_r^T x(t) \end{aligned} \quad (1.2)$$

Let us introduce a new state equation

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= C^T x(t) \end{aligned} \quad (1.3)$$

in which

$$x(t) = \begin{bmatrix} x_r(t) \\ x_m(t) \end{bmatrix} \quad A = \begin{bmatrix} A_r & 0 \\ 0 & A_m \end{bmatrix} \quad B = \begin{bmatrix} B_r \\ B_m \end{bmatrix} \quad C = \begin{bmatrix} C_r \\ -C_m \end{bmatrix} \quad (1.4)$$

where in (1.1)-(1.4) $u(t)$ and $y(t)$ are input and output respectively, $x(t)$ is state vector, A, B, C are real matrices of corresponding dimensions.

2. Signals maximizing the integral square error

2.1. Existence and attainability of signals with two constraints

Let us assume that U is the set of signals $u(t)$ segmentarily C^1 over the interval $[0, T]$, and the error $y(t)$ of the measuring system relative to its standard is expressed by inner product

$$I(u) = \int_0^T [y(t)]^2 dt = (Ku, Ku) \quad u \in U \quad (2.1)$$

where

$$Ku = y(t) = \int_0^t k(t - \tau)u(\tau) d\tau \quad (2.2)$$

and

$$k(t) = C^T e^{At} B \quad (2.3)$$

Let us consider the signal $h \in U$ and let the following condition be fulfilled

$$\forall 0 < b < c < T \quad \exists h \in U : \text{supp } h \subset [b, c] \quad (2.4)$$

and positive square functional

$$I(h) > 0 \quad (2.5)$$

Let us define the following set A of signals with imposed constraints on magnitude a and rate of change ϑ

$$A := \{u(t) \in U : |u(t)| \leq a, |\dot{u}_+(t)| \leq \vartheta, |\dot{u}_-(t)| \leq \vartheta, \quad t \in [0, T]\} \quad (2.6)$$

where $\dot{u}_+(t)$ and $\dot{u}_-(t)$ are increasing and decreasing derivative of $u(t)$ respectively.

Let $u_0(t) \in A$ fulfils the condition

$$I(u_0) = \sup \{I(u) : u \in A\} \quad (2.7)$$

Theorem

$$\forall t \in [0, T], \quad |u_0(t)| = a \quad \text{or} \quad |\dot{u}_{0+}(t)| = \vartheta \quad \text{or} \quad |\dot{u}_{0-}(t)| = \vartheta \quad (2.8)$$

Proof

Suppose that (2.8) is not true. Then

$$\exists \varepsilon > 0, \quad \exists 0 < b < c < T \quad (2.9)$$

such that

$$|u_0(t)| \leq a - \varepsilon, \quad |\dot{u}_{0+}(t)| \leq \vartheta - \varepsilon, \quad |\dot{u}_{0-}(t)| \leq \vartheta - \varepsilon, \quad t \in (b, c) \quad (2.10)$$

Let us choose h according to (2.4)

$$\text{supp } h \subset [b, c], \quad I(h) > 0 \quad (2.11)$$

then for small $d \in \mathcal{R}$, say $d \in (-\delta, \delta)$ is

$$u_0 + dh \in A, \quad \forall d \in (-\delta, \delta) \quad (2.12)$$

and from the optimum condition $u_0(t)$ it results that

$$I(u_0) \geq I(u_0 + dh) \quad (2.13)$$

hence

$$I(u_0) \geq I(u_0) + d^2 I(h) + 2d(Ku_0, Kh), \quad d \in (-\delta, \delta) \quad (2.14)$$

and

$$0 \geq d^2 I(h) + 2d(Ku_0, Kh), \quad d \in (-\delta, \delta) \quad (2.15)$$

However, the last inequality will never be fulfilled for $I(h) > 0, d \in (-\delta, \delta)$. So, from this contradiction it easily results that $I(u_0)$ can fulfil condition (2.7) only if input signal $u_0(t)$ reaches one of the constraints given in (2.8).

Corollary

The proof presented above reduces the shape of the $u_0(t)$ signals to triangles or trapezoids if constraints on magnitude and rate of change are imposed simultaneously. It means that signals $u_0(t)$ can only take the form of triangles with the slope inclination $|\dot{u}_{0+}(t)| = \vartheta$ or $|\dot{u}_{0+}(t)| = -\vartheta$ or of trapezoids with the slopes $|\dot{u}_{0+}(t)| = \vartheta$ and $|\dot{u}_{0+}(t)| = -\vartheta$ and a magnitude of a .

It seems however to be impossible to find an analytical solution with respect to the shapes of the $u_0(t)$ and to the formula describing the maximum error effected by them.

Carrying out the proof in an identical way, it can be shown that if only one of the constraints is imposed to the signal, either of magnitude a or of the rate of change ϑ , then the functional $I(u_0)$ reaches maximum if the signal reaches this constraint over the interval $[0, T]$.

If only a magnitude constraint is imposed on the $u_0(t)$ signal then it is of “bang-bang” type and it is possible to determine its switching moments. Below we will present analytically solution for determining of such a signal.

2.2. Signals constraint on magnitude

If only constraint of magnitude a is applied to the input signal then $u_0(t)$ has the form of a “bang-bang” with the magnitude a , and the problem is limited to determining its switching instants only. In order to determine these switching let us consider the equation (2.1) which can be presented as follow

$$I(u) = (Ku, Ku) = (K^*Ku, u) \quad (2.16)$$

where operator K^* is conjugate to K

$$(K^*Ku, u) = \int_t^T k(\tau - t) \left(\int_0^\tau k(\tau - \nu)u(\nu) d\nu \right) d\tau \quad (2.17)$$

Let the signals $u \in U$ be limited in magnitude

$$|u(t)| \leq a, \quad 0 < a \leq 1 \quad (2.18)$$

From the condition of optimality (2.7) it results that

$$\left(\frac{\partial I(u)}{\partial u} \Big|_{u_0}, u - u_0 \right) \leq 0 \quad (2.19)$$

Having computed the derivative $\frac{\partial I(u)}{\partial u} \Big|_{u_0}$, considering (2.16) and performing simple transformations (2.19), yields

$$(K^*Ku_0, u) \leq (K^*Ku_0, u_0) \quad (2.20)$$

in which the right-hand side presents the maximum.

Left hand side of formula (2.20) reaches maximum, making both sides equal if a signal with a maximum permissible magnitude a

$$|u_0(t)| = a \quad (2.21)$$

has the form

$$u(t) = u_0(t) = \text{sign} [K^*Ku_0(t)] \quad (2.22)$$

After considering (2.17) we finally obtain

$$u_0(t) = a \cdot \text{sign} \left[\int_t^T k(\tau - t) \left(\int_0^\tau k(\tau - \nu)u_0(\nu) d\nu \right) d\tau \right] \quad (2.23)$$

The maximum value $\max I(u)$ generated by the signal $u_0(t)$ is

$$I(u_0) = \int_0^T |K^*Ku_0(t)| dt = a^2 \int_0^T \left| \int_t^T k(\tau - t) \left(\int_0^\tau k(\tau - \nu)u_0(\nu) d\nu \right) d\tau \right| dt \quad (2.24)$$

2.3. Algorithm for determining of $u_0(t)$ signal

From formula (2.23) it results that $u_0(t)$ is a signal of the “bang-bang” type with maximum magnitude assuming the value of $a = +1$ or $a = -1$ by virtue of (2.21) and with the switching instants t_1, t_2, \dots, t_n corresponding to the consecutive $i = 1, 2, \dots, n$ zeros of the function occurring under the *sign* mark in formula (2.23). In order to determine these instants, let us assume that the first switching of the signal $u_0(t)$ occurs from $+1$ to -1 . This means that in the first time interval for $0 < t < t_1$ signal $u_0(t) = +1$. Let us also assume that we will search for n switchings over interval $[0, T]$. On the basis of formula (2.23) we can write n equations with t_1, t_2, \dots, t_n as variables for those assumptions. It can be easily seen that the equations are described by the following relationship

$$\sum_{l=i}^n \int_{t_l}^{t_{l+1}} k(\tau - t_i) \left(\sum_{m=0}^l (-1)^m \int_{t_m}^{t_{m+1}} k(\tau - \nu) d\nu \right) d\tau = 0 \quad i = 1, 2, \dots, n \quad (2.25)$$

where $t_0 = 0$, $t_{n+1} = T$, $t_{m+1} = \tau$ for $m = l$, n - number of switchings. Solution of system equations (2.25) with respect to t_1, t_2, \dots, t_n gives the sought switchings instants of the signal $u_0(t)$. Between those instants, depending on the interval $t < t_1$, $t_1 \leq t < t_2$, $\dots, t \geq t_n$, function $K^*Ku_0(t, t_1, \dots, t_n)$ in (2.24) is determined by the system of $n + 1$ following relationships

$$K^*Ku_0(t, t_1, t_2, \dots, t_n) = \sum_{l=i-1}^n \int_{t_l}^{t_{l+1}} k(\tau - t) \left(\sum_{m=0}^l (-1)^m \int_{t_m}^{t_{m+1}} k(\tau - \nu) d\nu \right) d\tau \quad i = 1, 2, \dots, n + 1 \quad (2.26)$$

where $t_0 = 0$, $t_{n+1} = T$, $t_l = t$ for $l = i - 1$, $t_{m+1} = \tau$ for $m = l$, n - number of switchings.

The value $I(u_0)$ is determined by the sum of modules, which is determined by formula (2.26) over all $n + 1$ intervals

$$I(u_0) = \sum_{i=1}^{n+1} \left| \sum_{l=i-1}^n \int_{t_l}^{t_{l+1}} k(\tau - t) \left(\sum_{m=0}^l (-1)^m \int_{t_m}^{t_{m+1}} k(\tau - \nu) d\nu \right) d\tau \right| \quad (2.27)$$

Exemplary equations for $n = 3$ switching instants in t_1, t_2 and t_3 resulting from formulae (2.25) and (2.26) are as follow:

From (2.25) we have three equations

$$\begin{aligned} & \int_{t_1}^{t_2} k(\tau - t_1) \left[\int_0^{t_1} k(\tau - \nu) d\nu - \int_{t_1}^{\tau} k(\tau - \nu) d\nu \right] d\tau \\ & + \int_{t_2}^{t_3} k(\tau - t_1) \left[\int_0^{t_1} k(\tau - \nu) d\nu - \int_{t_1}^{t_2} k(\tau - \nu) d\nu + \int_{t_2}^{\tau} k(\tau - \nu) d\nu \right] d\tau \\ & + \int_{t_3}^T k(\tau - t_1) \left[\int_0^{t_1} k(\tau - \nu) d\nu - \int_{t_1}^{t_2} k(\tau - \nu) d\nu + \int_{t_2}^{t_3} k(\tau - \nu) d\nu - \int_{t_3}^{\tau} k(\tau - \nu) d\nu \right] d\tau \\ & = 0 \end{aligned} \quad (2.28)$$

$$\begin{aligned} & \int_{t_2}^{t_3} k(\tau - t_2) \left[\int_0^{t_1} k(\tau - \nu) d\nu - \int_{t_1}^{t_2} k(\tau - \nu) d\nu + \int_{t_2}^{\tau} k(\tau - \nu) d\nu \right] d\tau \\ & + \int_{t_3}^T k(\tau - t_2) \left[\int_0^{t_1} k(\tau - \nu) d\nu - \int_{t_1}^{t_2} k(\tau - \nu) d\nu + \int_{t_2}^{t_3} k(\tau - \nu) d\nu - \int_{t_3}^{\tau} k(\tau - \nu) d\nu \right] d\tau \\ & = 0 \end{aligned} \quad (2.29)$$

$$\begin{aligned} & \int_{t_3}^T k(\tau - t_3) \left[\int_0^{t_1} k(\tau - \nu) d\nu - \int_{t_1}^{t_2} k(\tau - \nu) d\nu + \int_{t_2}^{t_3} k(\tau - \nu) d\nu - \int_{t_3}^{\tau} k(\tau - \nu) d\nu \right] d\tau \\ & = 0 \end{aligned} \quad (2.30)$$

and we have four equations resulting from (2.26):

for $t < t_1$

$$\begin{aligned}
K^*Ku_0(t, t_1, t_2, t_3) &= \\
&= \int_t^{t_1} k(\tau - t) \left[\int_0^\tau k(\tau - \nu) d\nu \right] d\tau + \int_{t_1}^{t_2} k(\tau - t) \left[\int_0^{t_1} k(\tau - \nu) d\nu - \int_{t_1}^\tau k(\tau - \nu) d\nu \right] d\tau \\
&+ \int_{t_2}^{t_3} k(\tau - t) \left[\int_0^{t_1} k(\tau - \nu) d\nu - \int_{t_1}^{t_2} k(\tau - \nu) d\nu + \int_{t_2}^\tau k(\tau - \nu) d\nu \right] d\tau \\
&+ \int_{t_3}^T k(\tau - t) \left[\int_0^{t_1} k(\tau - \nu) d\nu - \int_{t_1}^{t_2} k(\tau - \nu) d\nu + \int_{t_2}^{t_3} k(\tau - \nu) d\nu - \int_{t_3}^\tau k(\tau - \nu) d\nu \right] d\tau
\end{aligned} \tag{2.31}$$

for $t_1 \leq t < t_2$

$$\begin{aligned}
K^*Ku_0(t, t_1, t_2, t_3) &= \\
&\int_t^{t_2} k(\tau - t) \left[\int_0^{t_1} k(\tau - \nu) d\nu - \int_{t_1}^\tau k(\tau - \nu) d\nu \right] d\tau \\
&+ \int_{t_2}^{t_3} k(\tau - t) \left[\int_0^{t_1} k(\tau - \nu) d\nu - \int_{t_1}^{t_2} k(\tau - \nu) d\nu + \int_{t_2}^\tau k(\tau - \nu) d\nu \right] d\tau \\
&+ \int_{t_3}^T k(\tau - t) \left[\int_0^{t_1} k(\tau - \nu) d\nu - \int_{t_1}^{t_2} k(\tau - \nu) d\nu + \int_{t_2}^{t_3} k(\tau - \nu) d\nu - \int_{t_3}^\tau k(\tau - \nu) d\nu \right] d\tau
\end{aligned} \tag{2.32}$$

for $t_2 \leq t < t_3$

$$\begin{aligned}
K^*Ku_0(t, t_1, t_2, t_3) &= \\
&= \int_t^{t_3} k(\tau - t) \left[\int_0^{t_1} k(\tau - \nu) d\nu - \int_{t_1}^{t_2} k(\tau - \nu) d\nu + \int_{t_2}^\tau k(\tau - \nu) d\nu \right] d\tau \\
&+ \int_{t_3}^T k(\tau - t) \left[\int_0^{t_1} k(\tau - \nu) d\nu - \int_{t_1}^{t_2} k(\tau - \nu) d\nu + \int_{t_2}^{t_3} k(\tau - \nu) d\nu - \int_{t_3}^\tau k(\tau - \nu) d\nu \right] d\tau
\end{aligned} \tag{2.33}$$

for $t > t_3$

$$\int_t^T k(\tau - t) \left[\int_0^{t_1} k(\tau - \nu) d\nu - \int_{t_1}^{t_2} k(\tau - \nu) d\nu + \int_{t_2}^{t_3} k(\tau - \nu) d\nu - \int_{t_3}^\tau k(\tau - \nu) d\nu \right] d\tau \tag{2.34}$$

For different number of n switchings in t_1, t_2, \dots, t_n , we can set up a relevant system of equations in a similar way. The procedure of searching for the optimum number of n switchings commences with the assumption $i = 1$, the solution of equation (2.25) with respect to t_1 , and checking the value $I(u_0)$ (2.27) corresponding to the obtained solution. Next the procedure is repeated for $i = 2, 3, \dots$. In this way, the upper value of n is not given in advance, but it is being consecutively increased until the $I(u_0)$ resulting from formula (2.27) reaches a maximum. Such a situation occurs when the value $I(u_0)$ reached for $n + 1$ switchings is not higher than the value of this error corresponding to n switchings, and a further increase in the number of switchings cannot increase it any further. In consequence the process of searching for the optimum number of switchings will conclude at this value of n .

3. Signals maximizing the absolute value of error

3.1. Signals constraint in magnitude

In order to determine signal maximizing the value of absolute error let us take into consideration a convolution integral (2.2)

$$y(t) = \int_0^t k(t - \tau)u(\tau) d\tau \quad t \in [0, T] \quad (3.1)$$

from which it directly results that maximum $|y(t)|$ occurs for $t = T$

$$\max |y(t)| = y(T) = \int_0^T k(T - \tau)u(\tau) d\tau \quad (3.2)$$

if

$$u(\tau) = u_0(\tau) = a \cdot \text{sign } k(T - \tau) \quad (3.3)$$

Changing in (3.2) τ for t we can write

$$|y(t)| = y(T) = \int_0^T k(T - t)u(t) dt \quad (3.4)$$

and $u_0(t)$ maximizing (3.4) has now the form

$$u_0(t) = a \cdot \text{sign } k(T - t) \quad (3.5)$$

which, for t_n switchings in t_1, t_2, \dots, t_n and assumption that the first switching occurs from $+a$ to $-a$, can be determined by means of the following relationship

$$u_0(t_1, t_2, \dots, t_n) = a \cdot \sum_{i=0}^n (-1)^i \int_{t_i}^{t_{i+1}} k(T - t) dt \quad (3.6)$$

where: $t_0 = 0$; $t_{i+1} = T$ for $i = n$; n - number of switchings.

Substitution of (3.5) into (3.4) gives finally

$$\max |y(t)| = y(T) = a \cdot \int_0^T |k(T - t)| dt = a \int_0^T |k(t)| dt \quad (3.7)$$

which is not difficult to compute.

3.2. Shape of signals with two constraints

Let us present signal $u(t)$ in (3.1) by means of the integral

$$u(t) = \int_0^t \varphi(\tau) d\tau \quad (3.8)$$

The error (3.4) can be written now in the following form

$$y(T) = \int_0^T k(T - t) \int_0^t \varphi(\tau) d\tau dt \quad (3.9)$$

and constraints (2.6) referring to $u(t)$ for function are now as follow

$$\left| \int_0^t \varphi(\tau) d\tau \right| = |u(t)| \leq a \quad (3.10)$$

and

$$|\varphi(\tau)| = |\dot{u}(t)| \leq \vartheta \quad (3.11)$$

Changing integration order in (3.9) we have

$$y(T) = \int_0^T \varphi(\tau) \int_t^T k(T-t) dt d\tau \quad (3.12)$$

which, after replacing τ for t , gives finally

$$y(T) = \int_0^T \varphi(t) \int_t^T k(T-\tau) d\tau dt \quad (3.13)$$

From (3.13) it can be easily seen that $\varphi(t)$ maximizing $y(T)$ is a signal with maximum magnitude $\varphi(t) = \pm\vartheta$ by virtue of formula (3.11) in the intervals determined by switching moments resulting from the following relationship

$$\varphi(t) = \text{sign} \int_t^T k(T-\tau) d\tau \quad (3.14)$$

or it has a zero value of magnitude $\varphi(t) = 0$ in these subintervals, for which between the switching moments resulting from (3.14) is

$$\left| \int_0^t \varphi(\tau) d\tau \right| > a \quad (3.15)$$

Using equations (3.13)-(3.15) we can determine signal $u(t) = u_0(t)$ in the following steps: In the first step “bang-bang” functions $f_1(t)$ of magnitude $\pm\vartheta$ are determined with switching moments resulting from (3.14) - Fig.1

$$\begin{aligned} f_1(t) &= +\vartheta & \text{if } \varphi(t) > 0 \\ f_1(t) &= -\vartheta & \text{if } \varphi(t) < 0 \end{aligned} \quad (3.16)$$

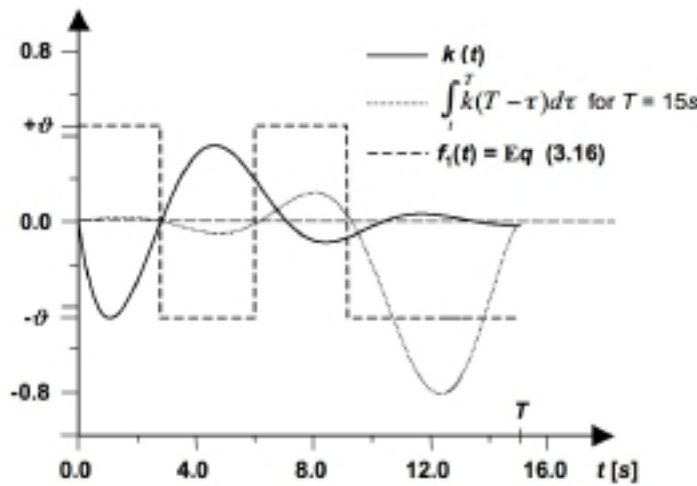


Fig.1. Exemplary functions $k(t)$, $f_1(t)$ and $\int_t^T k(T-\tau) d\tau$

In the second step integrating function $f_1(t)$ we receive function $f_2(t)$ - Fig. 2. Denoting switching moments of $f_1(t)$ by t_1, t_2, \dots, t_n and the number of switchings by n we have:

for: $t \leq t_1 ; n = 1$

$$f_2(t) = \vartheta \cdot t \quad (3.17)$$

for: $t_1 \leq t < t_2 ; n = 2,$

$$f_2(t) = \vartheta \cdot t_1 - \vartheta \cdot (t - t_1) \quad (3.18)$$

for: $t_i \leq t \leq t_{i+1}$; $i = 2, 3, \dots, n$; $t_{n+1} = T$

$$f_2(t) = \vartheta \cdot t_1 + \vartheta \cdot \left| \sum_{j=2}^i (-1)^{j-1} \cdot (t_j - t_{j-1}) \right| + (-1)^i \cdot \vartheta \cdot (t - t_i) \quad (3.19)$$

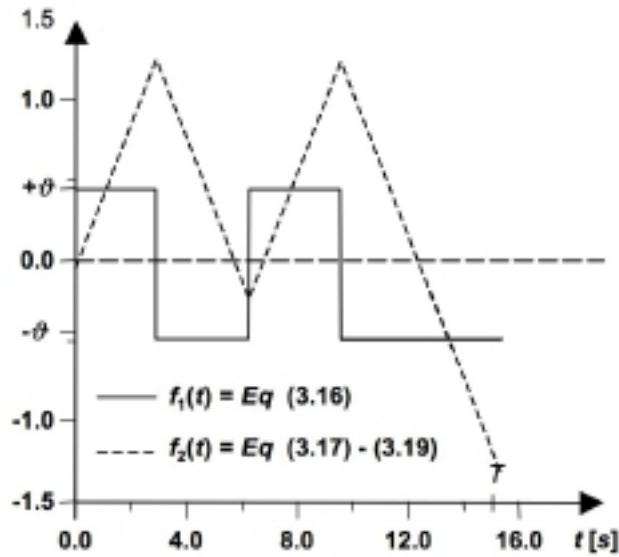


Fig.2. Functions $f_1(t)$ Eq.(3.16) and $f_2(t)$ Eqs. (3.17)-(3.19)

In the next step on the base of $f_2(t)$ function we determine function $f_3(t)$ which is :

$$\begin{aligned} f_3(t) &= \pm\vartheta & \text{if } |f_2(t)| \leq a \\ f_3(t) &= 0 & \text{if } |f_2(t)| > a \end{aligned} \quad (3.20)$$

Finally integrating $f_3(t)$ we receive signal $u(t) = u_0(t)$ - Fig. 3 which was searched for.

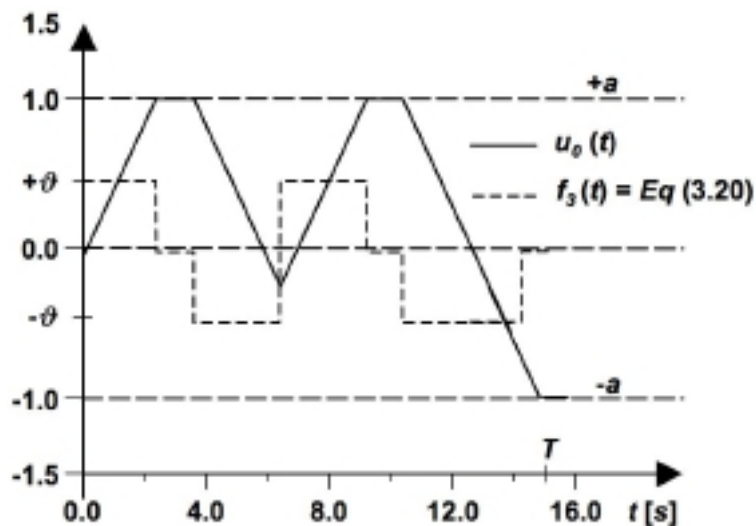


Fig. 3. Function $f_3(t)$ Eq.(3.20) and signal $u_0(t)$

In this way, the signal has a triangular shape with the slope of $\pm\vartheta$ in the intervals in which $f_3(t) = \pm\vartheta$, or it presents a constant value of magnitude $\pm a$ in these intervals in which $f_3(t) = 0$.

From (3.17) – (3.20) it clearly follows that for m switchings moments of function $f_3(t)$ the value of error (3.4) is described by the following equations:

for $n = 1$

$$y(T) = \frac{h_1}{t_1} \cdot \int_0^{t_1} k(T - \tau) \cdot \tau d\tau + \frac{h_T - h_1}{T - t_1} \cdot \int_{t_1}^T k(T - \tau) \cdot (\tau - t_1) d\tau + h_1 \cdot \int_{t_1}^T k(T - \tau) d\tau \quad (3.21)$$

for $n \geq 2$

$$\begin{aligned} y(T) = & \frac{h_1}{t_1} \cdot \int_0^{t_1} k(T - \tau) \cdot \tau d\tau + \\ & + \sum_{i=2}^n \left[\frac{h_i - h_{i-1}}{t_i - t_{i-1}} \cdot \int_{t_{j-1}}^{t_j} k(T - \tau) \cdot (\tau - t_{i-1}) d\tau + h_{i-1} \cdot \int_{t_{j-1}}^{t_j} k(T - \tau) d\tau \right] + \\ & + \frac{h_T - h_n}{T - t_n} \cdot \int_{t_n}^T k(T - \tau) \cdot (\tau - t_n) d\tau + h_n \cdot \int_{t_n}^T k(T - \tau) d\tau \end{aligned} \quad (3.22)$$

where in n - number of switchings of $u_0(t)$, t_i - switching moment, $h_i = u_0(t_i)$, $h_T = u_0(T)$.

Conclusion

The solutions presented in this paper can be applied to the calibration of measuring systems intended for the measurement of dynamic signals, particularly for systems for which the input signals are unknown and varying in time. The error determined using signals of the form $u_0(t)$ is analogous to the maximum error commonly determined when calibrating instruments dedicated to static measurement. The shapes of $u_0(t)$ signal presented in this paper and the resulting formulae describing the maximum error refer to both integral square error and absolute error. Solutions for the two types of error are very different. This means that for each type of error, constraints imposed on input signal and type of mathematical model of system and its standard the problem of $u_0(t)$ signal determination must be separately and individually considered. In all cases however it is most important to prove mathematically that a signal which maximizes the chosen error exists as well as to determine the space in which this signal is available. In the case in which an analytical solution with respect to the shape of $u_0(t)$ signal and to the formula describing the maximum error effected by it is impossible to determine, various computer programs can be used in order to solve these problems, but it is essential to maximally limit the space in which the possible solutions are searched for. This significantly increases the likelihood of obtaining the correct solution, and considerably reduces the computation time.

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